The Null-Killing Surfaces of the Kerr and Kerr-Newman Solutions of the Einstein Field Equations

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ABSTRACT

The null Killing surfaces for the Kerr and Kerr-Newman solutions to the Einstein field equations are discussed, derived and plotted. For some parameters of the mass *m*, the angular momentum per unit mass *a*, and the charge *e*, the plots show some unusual features, particularly of the Kerr-Newman metric.

The Kerr-Newman Metric

The Kerr-Newman metric in Kerr-Schild coordinates is given by

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m\rho^3 - e^2\rho^2}{\rho^4 + a^2z^2}\left(k_\mu dx^\mu\right)^2, \tag{1}$$

where e is the charge, m the mass, a the angular momentum per unit mass, and the surfaces of constant ρ are confocal ellipsoids of revolution, the equation for which is derived from the defining relations for oblate spheroidal coordinates:

$$\begin{aligned} x &= a\ cosh\xi\ cos\eta\ cos\phi \\ y &= a\ cosh\xi\ cos\eta\ sin\phi \\ z &= a\ sinh\xi\ sin\eta. \end{aligned} \tag{2}$$

The null vector field k_μ is given by

$$k_\mu dx^\mu = dt + \frac{z}{\rho}dz + \frac{\rho}{\rho^2 + a^2}(xdx + ydy) + \frac{a}{\rho^2 + a^2}(xdy - ydx). \tag{3}$$

Note that if $\rho = e^2/2m$ the metric of Eq. (1) becomes that of 4-dimensional Euclidean space. The metric given by Eq. (1) has a ring singularity located at $R \coloneqq (x^2 + y^2)^{\frac{1}{2}} = a$ and $z = 0$ (where $\rho = 0$). The definition of R is important to remember in what follows.

From Eqs. (2) one can then compute

$$\frac{x^2 + y^2}{a^2\ cosh^2\xi} + \frac{z^2}{a^2\ sinh^2\xi} = 1. \tag{4}$$

Setting $\rho^2 = a^2 \, sinh^2\xi$ results in

$$\frac{x^2 + y^2}{\rho^2 + a^2} + \frac{z^2}{\rho^2} = 1. \tag{5}$$

ρ in the metric given by Eq. (1) is implicitly determined by Eq. (5) up to a sign.

A null Killing surface is also known as a Killing horizon. It is where a Killing vector changes from time-like to space-like or visa-versa when crossing the surface. The null Killing surfaces for the Kerr-Newman metric are obtained by setting the g_{00} component of the metric equal to zero. This results in the equation,

$$\rho^4 - 2m\rho^3 + e^2\rho^2 + a^2z^2 = 0. \tag{6}$$

To obtain the solutions to this equation, Eq. (5) is solved for *r* in terms *R* and *z* and one of the four solutions is substituted into Eq. (6). There are nine very long solutions to the resulting equation: one zero, four imaginary and four that correspond to pieces of the null Killing surface.

For comparison, the null Killing surfaces for the Kerr metric, where *e* = 0, have been given by Marsh.[1] Figures (1) and (2) from that work show examples for different values of the parameters *a* and *m*. The azimuthal angle has only been partially plotted in order to show the interior features.

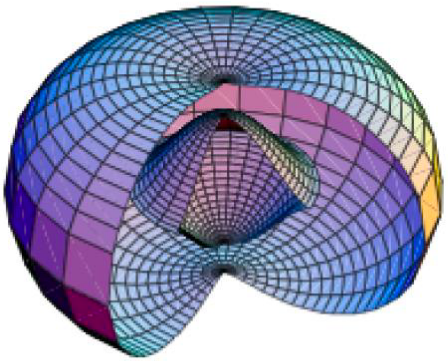


Figure 1. The Kerr null Killing surfaces for m =1.02 a. The ring singularity is at the cusp of the inner surface.

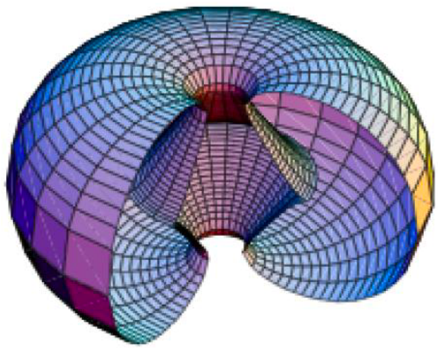


Figure 2 The Kerr null Killing surfaces for $a > m$. Here m = 0.98 a.

These null Killing surfaces are found by setting g_{00} = 0 in the Kerr metric. Explicitly, Eq. (6) is solved for z, and e is then set equal to zero followed by the null surfaces for the Kerr solution being plotted. For the Kerr-Newman metric, Eq. 11 of Marsh gives g_{00} as

$$g_{00} = \frac{x^2 + y^2 - a^2 - 2m(x^2 + y^2 - a^2)^{\frac{1}{2}} + e^2}{a^2 - (x^2 + y^2)}. \tag{7}$$

Before proceeding it is worth looking at the nature of the disk circled by the ring singularity in the Kerr solution. This space is flat and has the character of a quadratic branch point in the complex plane; that is, if one passes through the surface form above the coordinate labeling the oblate spheroidal surfaces of constant r is negative. The Kerr solution in the negative r region is identical in structure to the positive r part with m being replaced by its negative, which then causes g_{00} to change sign in this region.

The geometry around the disk circled by the ring singularity for the Kerr-Newman solution is quite different from that of the Kerr metric. The line element for the Kerr-Newman solution in Kerr-Schild coordinates with the mostly minuses signature is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{r^2(2Mr - e^2)}{r^4 + a^2z^2}\left[\frac{[r(x\,dx + y\,dy) - a(x\,dy - y\,dx)]}{r^2 + a^2} + \frac{z\,dz}{r} + dt\right]^2. \tag{8}$$

As can be seen from the term before the first set of brackets, the metric becomes that of flat Minkowski space for $r = e^2/2M$. This also true for negative r, since M changes sign for that region. This was noted by López[2], following Israel[3], when attempting to create a classical model of the spinning electron using the Kerr-Newman solution. The Kerr-Newman metric has no null Killing surfaces for $e^2 > m^2$ nor for $e^2 = m^2$ and $a > 0$. The parameters for the electron tells us that in this model there would be no null Killing surfaces.

The nature of g_{00} for the Kerr-Newman solution can be seen by plotting Eq. (7). The result is shown in Fig. 3.

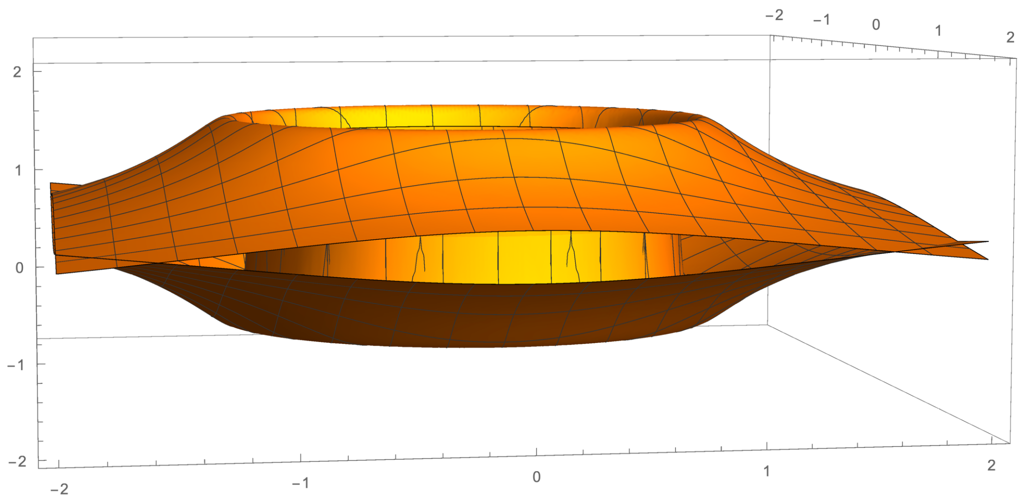


Figure 3. The surfaces of g_{00} for the Kerr-Newman solution. The parameters used for the figure are a = 1, e = 0.9, m = 1.34536, and z = 0. Note that the surfaces are outside the ring singularity and make no contact with it.

For the null Killing surfaces of the Kerr-Newman metric, Eq. (5) is again solved for ρ in terms of R and z and one of the four solutions substituted into Eq. (6). There are nine solutions to the resulting equation: one zero, four imaginary, and four that again correspond to pieces of the null Killing surfaces. In the case of the Kerr metric, when e is zero, there are only eight solutions. For certain choices of the parameters m, a, and e, either one or both of the null Killing surfaces may not exist, and the relationship of the ring singularity to these surfaces changes.

For $e = 0$, the null Killing surfaces of the Kerr-Newman solution are shown in Fig. 4.

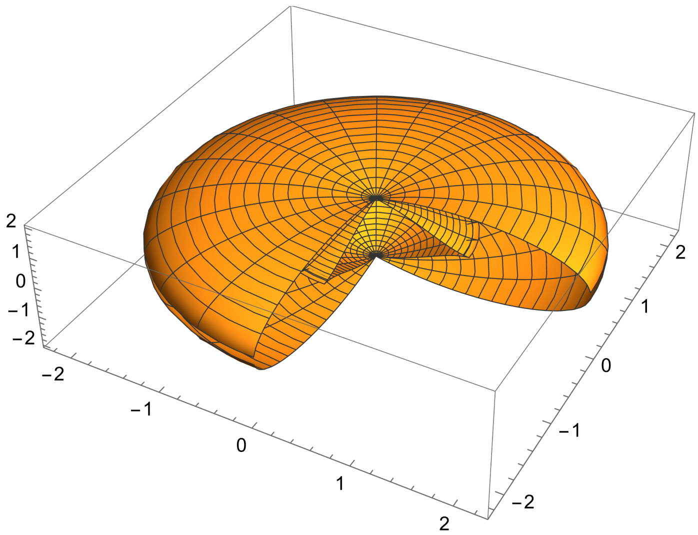


Figure 4. The Kerr-Newman null Killing surfaces for $e = 0$ and $a = m = 1$. The ring singularity for $a = 1$ borders the inner null surface as in the Kerr metric.

Note that the proportion of the height and width of Fig. (4), and in the following plots, reflect the "golden ratio" of 1.61803, which flattens the figures. The golden ratio is a constant the gives the limiting value of ratios of successive Fibonacci numbers. Because r takes both positive and negative values the metric is smooth everywhere away from the ring singularity. The space where r is negative is asymptotically flat. For $r < 0$, the azimuthal vector is timelike so that there are closed timelike curves.[4] These non-causal curves extend a small distance into the positive r region. Note that e and m respectively represent the charge and mass in the limit of large positive r. In the limit of large negative r, the mass and charge are $-m$ and $-e$.

With regard to the closed timelike curves of the Kerr-Newman solution for $r < 0$, there is a relevant theorem given by Geroch[5] that every compact geometry without boundary has closed timelike curves. The closed Friedmann model of the universe, topologically a 3-sphere having a compact spacelike surface, is an example.

For the parameters $e = 0.1$ and $a = m = 1$, the null Killing surfaces are shown in Fig. 5.

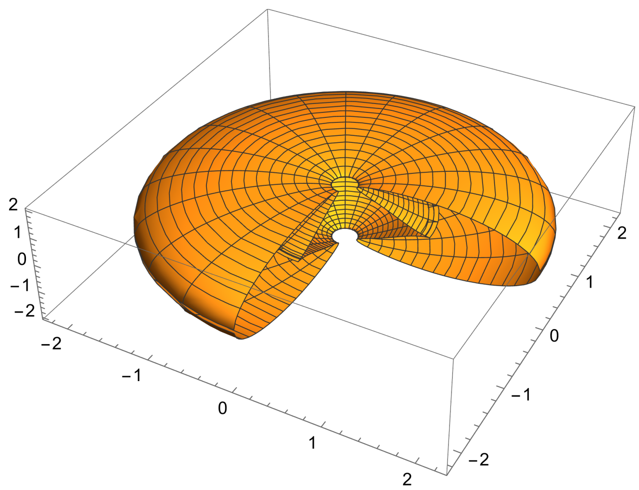


Figure 5. The Kerr-Newman null Killing surfaces for $e = 0.1$ and $a = m = 1$.

There are two important features of Fig. (5) to note: the first is that a non-zero value of e opens up the surfaces at the poles allowing passage into the inner null surface, as is the case for the Kerr metric with $a > m$; and the second is that the inner surface does not terminate at the ring singularity located at $r = 1$, unlike the Kerr metric, but at a somewhat greater value of r. This means that the ring singularity is reachable from outside the surfaces. What cannot easily be seen in Fig. 5 is that there is a gap at the equator. This is shown in Fig. 6.

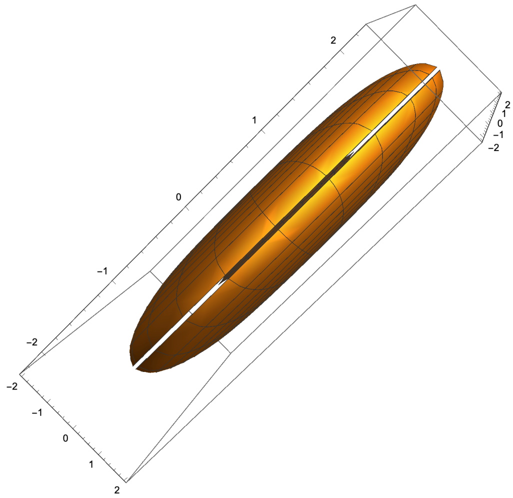
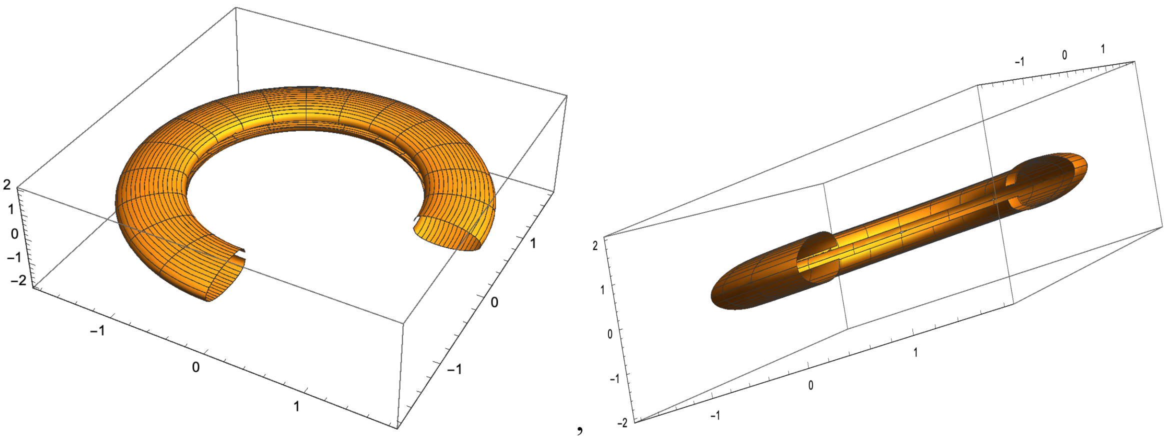


Figure 6. An equatorial view of the Kerr-Newman null Killing surfaces for e = 0.1 and a = m = 1 showing the gap barely visible in Fig. 5.

As the value of e increases, the gap in the null surfaces decreases, and when $a^2 + e^2 > m^2$ and $m > a > e$ the null surface becomes a toroid. This is shown in Fig. 7(a) and (b).



(a) (b)

Figure 7. the Kerr-Newman null Killing surfaces for e = 0.9, a = 1, and m = 1.02. The ring singularity is outside the toroidal surface.

The gap seen in the inner part of the toroid shown in Fig. (7) measures ~0.4. As pointed out above, the metric becomes that of flat Minkowski space for $r = e^2/2m$. For the values of the parameters used to plot Fig.7, $e^2/2m$ ~0.4, which is what the gap measures in the figure. Here is another relevant figure from the paper by Marsh:

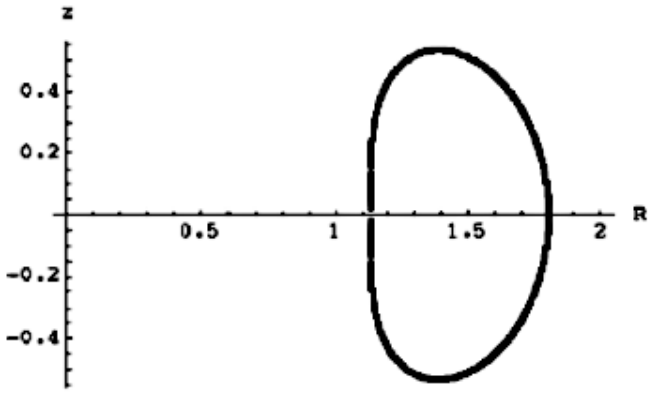


Figure 8. The null Killing surface for the Kerr-Newman solution with m =1.02, a = 1, and e = 0.9 is obtained by rotating this figure about the z-axis. The time-like Killing vector becomes space-like within the toroid. The ring singularity is located at R = 1. While it is not obvious from the figure, there is a gap where graph crosses the R-axis at R~1.12. The gap is far more obvious in Fig. 7.

Astrophysical Applications of the Kerr and Kerr-Newman Metrics

Besides the Schwarzschild and Reissner-Nordström metrics, two of the most important metrics in general relativity are the Kerr and Kerr-Newman solutions. The first describes the gravitational field or a rotating body and its applicability to the exterior of massive bodies has been well confirmed by observation. The Kerr-Newman solution describes the gravitational and electromagnetic fields of a rotating and charged mass. For astrophysical purposes, however, this solution has had no confirmation by observations.

In order to describe the entire spacetime of either solution, the exterior solutions must be matched to an interior solution. Unfortunately, there are no known non-singular interior solutions for these metrics. Nonetheless, the Kerr solution has been found to have great astrophysical applicability.

An additional problem exists for the Kerr-Newman solution. Whether or not the universe as a whole is charged or not, it is believed that conservation of charge would still apply. This implies that massive bodies described by the Kerr-Newman solution must be created in pairs having opposite electric charge. Even if this could occur, the charges would likely be neutralized by surrounding ionized gas.

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